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Uniformly valid analytical solution to the problem of a decaying shock wave

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An explicit representation of an analytical solution to the problem of decay of a plane shock wave of arbitrary strength is proposed. The solution satisfies the basic equations exactly. The approximation lies in the (approximate) satisfaction of two of the Rankine–Hugoniot conditions. The error incurred is shown to be very small even for strong shocks. This solution analyses the interaction of a shock of arbitrary strength with a centred simple wave overtaking it, and describes a complete history of decay with a remarkable accuracy even for strong shocks. For a weak shock, the limiting law of motion obtained from the solution is shown to be in complete agreement with the Friedrichs theory. The propagation law of the non-uniform shock wave is determined, and the equations for shock and particle paths in the (x, t)-plane are obtained. The analytic solution presented here is uniformly valid for the entire flow field behind the decaying shock wave.

1. Introduction

In the theory of one-dimensional unsteady gas motion there is an interesting but very complicated problem related to the decay of a plane shock wave when it interacts with a simple wave. An explicit solution of this problem involving a non-isentropic flow with an unknown moving boundary, a shock wave of arbitrary strength, has been judged so far as not feasible. The simplest case of a motion involving such an interaction arises when a piston moving with uniform velocity into a gas at rest suddenly stops and sends out a rarefaction wave to interact with the shock wave which races ahead of it. The details of the flow field resulting from this interaction in the (x, t)-plane are shown in figure 1. A piston moving along PO sends in front of it a uniform shock front (its path represented by the straight path PE) with constant states ahead of and behind it. The piston is suddenly stopped at O, leading to the emission of rarefaction waves, which overtake the shock from behind and render it non-uniform. When the uniform shock front PE is overtaken from behind by the leading expansion front OE of a forward-facing centred simple wave, its trajectory ceases to be a straight line. It is represented by the curved path EF in the figure. The front OE on impinging upon the shock sends a signal propagating back into the flow region along a receding Mach line. Thus, the flow field behind the shock front PEF can be divided into four regions. The region R_1 has a uniform flow. The region R_2 is spanned by an expansion fan of a centred simple wave. The region R_3, which results from the interaction of two simple waves, is described by the general solution of the isentropic equations of motion and is called the general wave region.

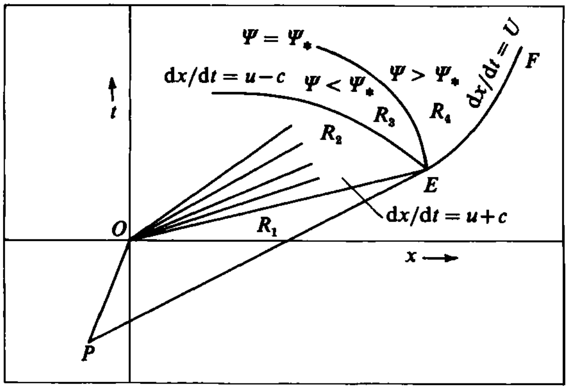

FIGURE 1. Interaction of a shock wave *PEF* with a centred simple expansion wave *OE*.

In region R_4, the gas motion is non-isentropic and is headed by a non-uniform shock advancing into a region at rest. Non-uniformity of the shock causes entropy variations, which are carried along the particle paths into the region behind the shock wave. These entropy variations make an analytic approach quite difficult. However, a considerable simplification results if the shock is weak so that the changes of entropy across it are negligible, i.e. if the flow behind the decaying shock can be assumed to be isentropic. In this case, the Riemann invariant is constant through the shock, and the (isentropic) simple wave solution satisfies the Rankine–Hugoniot (RH) conditions at the shock front to third order in shock strength. With this approximation, Friedrichs' (1948) theory provides a solution to this problem. Other methods by which the rise of entropy across the shock can be accounted for approximately have been given by Pillow (1949) and Lighthill (1950). Meyer (1960) and Meyer & Ho (1963) refer to an analytical approach, which can be used for describing approximately the *early* stages of decay of a shock of arbitrary strength. However, an analytical description of the complete history of decay of a shock of arbitrary strength does not seem to have appeared in the literature. Ardavan-Rhad (1970), by setting a limit on the strength of the shock, obtained an analytic solution which satisfies, at the shock of moderate strength, the kinematic boundary conditions exactly but the RH conditions approximately. However, the arbitrary function $h(\pi)$, which occurs in his solution, shows divergent behaviour as it is determined in the two separate ways: it is an increasing function of π if it is obtained from the integration of his (3.3), and a decreasing function of π if its form is obtained directly from the solution (see (3.4) of Ardavan-Rhad and our table 1). This discrepancy becomes strong as the shock strength increases. Ardavan-Rhad's solution would, therefore, become incompatible with the RH conditions in the high-Mach-number regime. This is borne out by his comments about the growth of the error in his approximate theory as the shock Mach number increases.

We present here an analytical solution to this shock–rarefaction interaction problem, which is not limited by the strength of the shock. The basic assumption in the form of our solution is that the particle velocity is a linear function of the spatial coordinate. A class of such flows has been considered by Pert (1980) (see the references to considerable previous work with linear velocity profiles in this paper). Pert refers to these particular flows as self-similar. This is a rather specialized definition of self-similarity, but numerical study of the given set of partial differential equations shows that a class of initial profiles do indeed approach the present 'self-similarity' form as time becomes large. The flows manifesting such behaviour include both expansion and compression of gas masses. Pert has shown that linear dependence of velocity on the spatial coordinate is equivalent to the condition that all state variables are separable functions of time and the Lagrangian variable. We, however, were led to the present solution by an earlier study of Steketee (1979), who considered 'homogeneous' solutions of non-homentropic flows in the Lagrangian mass coordinate and time. While our special solution satisfies the governing PDE's exactly, two of the RH conditions are met approximately; the kinematic shock condition and the sound speed at the shock are employed exactly in the treatment. However, numerical results show that the error of approximation in the other RH conditions is small even for strong shocks of pressure ratio as high as ten. The limiting law of motion of weak shocks is shown to be in full agreement with the Friedrichs theory.

The scheme of this paper is as follows. Section 2 contains the motivation for the non-isentropic solution and its subsequent derivation. The solution pertaining to region R_4 of non-isentropic flow is given in §3. Section 4 deals with the isentropic regions denominated as R_3, R_2 and R_1. The conclusions of this study are set forth in §5.

2. Derivation of a solution of the non-isentropic equations of motion

The Eulerian equations governing the unsteady one-dimensional motion of a fluid in the absence of transport effects are

$$\left.\begin{aligned} \rho(u_t+uu_x)+p_x &= 0, \\ \rho_t+u\rho_x+\rho u_x &= 0, \\ \eta_t+u\eta_x &= 0, \end{aligned}\right\} \tag{1}$$

where t denotes the time, x the position, ρ the density, u the particle velocity, p the pressure, and η the entropy per unit mass. The subscripts denote partial differentiation unless stated otherwise. When written in terms of the sound speed $c = (\gamma p/\rho)^{\frac{1}{2}}$, with γ as the ratio of specific heats c_p and c_v, the system (1) assumes the alternative form

$$c_t+uc_x+\tfrac{1}{2}(\gamma-1)\,cu_x = 0, \tag{2a}$$

$$u_t+uu_x+\frac{2cc_x}{(\gamma-1)} = \frac{c^2\eta_x}{c_p(\gamma-1)}. \tag{2b}$$

In order to describe some simple non-homentropic flows, we consider 'homogeneous' solutions in the Lagrangian mass coordinate h and time t. The system (1), when expressed in Lagrangian mass coordinate h and t, takes the form (Stanyukovich 1960)

$$u_t+p_h = 0, \tag{3a}$$

$$V_t-u_h = 0, \tag{3b}$$

$$pV^\gamma = \exp\left(\frac{\eta}{c_v}\right) = b(h), \tag{3c}$$

where $V = 1/\rho$ is the specific volume and $b(h)$ represents the entropy distribution. Equation (3a) represents the conservation of mass, (3b) the conservation of momentum and (3c) results from the conservation of energy and the equation of state for a perfect gas. Following a classical procedure of self-similar solutions, we write

$$V = h^r V_*(z), \quad p = h^q p_*(z), \quad u = h^w u_*(z), \tag{4}$$

where $z = t/h$ is a similarity variable and r, q and w are real constants.

Inserting (4) into (3), one finds that solutions of this kind are possible provided V, p and u are homogeneous functions of the same degree, say n, in h and t, so that $r = q = w = n$. Equations (3) then reduce to

$$V'_* + zu'_* - nu_* = 0, \tag{5a}$$

$$u'_* - zp'_* + np_* = 0, \tag{5b}$$

$$p_* V_*^\gamma h^{n(\gamma+1)} = b(h) = b_0 h^{(\gamma+1)n}, \tag{5c}$$

say, where a new constant b_0 is introduced, and a prime denotes differentiation with respect to z. It may be noted that the case $n = 0$ corresponds to a homentropic flow; for a non-homentropic flow, n is not zero.

An inspection of (5a) and (5b) leads to the following form of the solution:

$$V_* = V_0 z^m, \quad u_* = u_0 z^{m-1}, \quad p_* = p_0 z^{m-2}, \tag{6}$$

where V_0, u_0, p_0, m and b_0 are real constants given by

$$m = \frac{2}{\gamma+1}, \quad p_0 = \frac{(\gamma-1)\, u_0}{2\gamma + n\gamma + n}, \tag{7a, b}$$

$$V_0 = \tfrac{1}{2}(n\gamma + \gamma + n - 1)\, u_0, \tag{7c}$$

$$b_0 = \frac{2^{-\gamma} u_0^{\gamma+1} (\gamma-1)(n\gamma+\gamma+n-1)^\gamma}{n\gamma + 2\gamma + n}. \tag{7d}$$

Thus, from (3c) and (5c), one can deduce that

$$\eta = c_v\{\ln(b_0) + n(\gamma+1)\ln(h)\}. \tag{8}$$

Inspection of (3) shows that if u is a solution, then u + const. is also a solution. This invariance property of the velocity u allows a constant to be added to its form obtained above. Thus, one obtains the following class of 'homogeneous' solutions for the system (3):

$$V = V_0 h^{n-m} t^m, \quad p = p_0 h^{(n-m+2)} t^{m-2}, \quad u = u_0 h^{n-m+1} t^{m-1} + u_\infty, \tag{9a, b, c}$$

where the constant u_∞ may be regarded as the terminal velocity of all fluid elements. Several interesting properties of such homogeneous solutions in Lagrangian coordinates h and t have been discussed by Steketee (1979).

In order to obtain the corresponding solution in the (x, t)-plane one may use the transformation (Stanyukovich 1960)

$$x_h = V, \quad x_t = u,$$

which, on substitution of V and u from (9) and use of (7), leads to

$$x = \left(\frac{u_0}{m}\right) h^{n-m+1} t^m + u_\infty t. \tag{10}$$

Elimination of h from (9c) and (10) yields the following velocity distribution in the (x, t)-plane:

$$u(x, t) = m\left(\frac{x}{t}\right) + (1-m)\, u_\infty. \tag{11}$$

This (spatially) linear form of velocity distribution motivates us to seek a solution of the system (1) or (2) such that

$$u = x\Phi(t)+A(t). \tag{12}$$

When (12) is substituted in (2*a*), one obtains a linear partial differential equation in c, a general solution of which can be found to be

$$c = \mu^{\frac{1}{2}(\gamma-1)}[f(\Psi)]^{\frac{1}{2}}. \tag{13}$$

where

$$\mu = \exp\left(-\int \Phi(t)\,\mathrm{d}t\right), \tag{14}$$

and $f(\Psi)$ is an arbitrary function of

$$\Psi = \mu x - \int A(t)\,\mu(t)\,\mathrm{d}t. \tag{15}$$

It may be noted that Ψ remains constant along a particle path, $\mathrm{d}x/\mathrm{d}t = u$, and changes its value from one particle path to another. One may, therefore, regard Ψ as a Lagrangian coordinate. It may be noted that the entropy is also constant along a particle path, and is therefore a function of Ψ.

Substituting u and c from (12) and (13) into the momentum equation (2*b*), one finds, on using (15), that the former is satisfied provided the following conditions hold:

$$\dot{\Phi}(t)+\Phi^2(t) = B_1\mu^{\gamma+1}(t). \tag{16a}$$

$$\dot{A}(t)+A(t)\,\Phi(t) = -B_1\mu^{\gamma}(t)\int A(t)\,\mu(t)\,\mathrm{d}t, \tag{16b}$$

$$f(\Psi)\,\eta'(\Psi) = c_p f'(\Psi)+B_1(\gamma-1)\,c_p\,\Psi, \tag{16c}$$

where a prime and a dot denote derivatives with respect to Ψ and t, respectively.

Differentiating (16*b*) with respect to t and using (14) and (16*a*), we obtain

$$\ddot{A}(t)+(\gamma+1)\,\dot{A}(t)\,\Phi(t)+[2\dot{\Phi}(t)+(\gamma+1)\,\Phi^2(t)]\,A(t) = 0.$$

This equation, in two unknown functions, has a solution appropriate to the present problem, namely

$$A(t) = \alpha(t-t^*)^{-1}+\beta; \quad \Phi(t) = \frac{2}{(\gamma+1)\,(t-t^*)}, \tag{17}$$

where α, β and t^* are arbitrary constants.

Equations (16*a*, *b*) then yield

$$\mu(t) = (t-t^*)^{-2/(\gamma+1)}, \quad B_1 = -\frac{2(\gamma-1)}{(\gamma+1)^2}. \tag{18a, b}$$

Now, it follows from (13), (15) and (18*a*) that

$$c = (t-t^*)^{m-1}[f(\Psi)]^{\frac{1}{2}}, \tag{19}$$

$$x(t-t^*)^{-1} = \Psi(t-t^*)^{m-1}-\tfrac{1}{2}\alpha(\gamma+1)\,(t-t^*)^{-1}+\beta(1-m)^{-1}, \tag{20}$$

and so (12) can be written as

$$u = \left[\frac{2\Psi}{\gamma+1}\right](t-t^*)^{m-1}+\beta(1-m)^{-1}. \tag{21}$$

Equations (19)–(21) express the variables c, u and x in terms of the Lagrangian particle coordinate Ψ and time t. We may choose $t^* = 0$ without loss of generality.

We now identify the velocity distribution (21) with that of the special exact solution (9) of the non-homentropic case so that

$$u_\infty = \frac{\beta}{1-m}, \quad \Psi = \tfrac{1}{2}(\gamma+1)\,u_0 h^{n-m+1}. \tag{22a, b}$$

This identification refers to the case for which the velocity distribution (21), together with (22*b*), corresponds to a class of homogeneous flows described by (9); in view of (8) and (22*b*), the entropy distribution for such flows in the (x,t)-plane has the form

$$\eta = \eta_0 + 2Nc_p \ln(\Psi), \tag{23}$$

where $N = n/[(2-m)(n-m+1)]$, $\eta_0 = c_v \ln[b_0(m/u_0)^{2n/(m+mn-m^2)}]$, and $\Psi(x,t)$ is the same as in (20). We look for a particular solution in the (x,t)-plane, which may describe the decay behaviour of a plane shock due to interaction with a rarefaction wave, and for which the velocity and entropy distribution are given by (21) and (23). The motivation for the linear choice (12) leading to (21) and (23) has been given in §1. The integration of the remaining compatibility condition (16*c*) yields

$$f(\Psi) = \left[B_2 + 2(1-m)^2 \int \Psi \exp\left(-\frac{\eta}{c_p}\right) \mathrm{d}\Psi\right] \exp\left(\frac{\eta}{c_p}\right), \tag{24}$$

with B_2 as a constant of integration.

Eliminating η from (23) and (24) we obtain the form of the function $f(\Psi)$ as

$$f(\Psi) = \lambda^2\Psi^2 + B_3\Psi^{2N}, \tag{25}$$

where $B_3 = B_2\exp(\eta_0/c_p)$ and $\lambda = [(1-m)(2-m)(n-m+1)/(n-m+2)]^{\frac{1}{2}}$. We may note that a homentropic flow corresponds to the case $n = 0$ (i.e. $\lambda = 1-m$, $N = 0$). For a non-homentropic flow, $n \neq 0$; the constant N, for all admissible values of n, is found to be less than one.

In order to determine B_2, we match the homentropic flow in the region $0 < \Psi < \Psi_*$ with the non-homentropic flow in $\Psi > \Psi_*$, across the boundary (streamline) $\Psi = \Psi_* = \exp[(\eta_1-\eta_0)/(2Nc_p)]$, where η_1 and η_0 are, respectively, the values of entropy behind and ahead of the uniform shock *PE*. Thus, for $\Psi < \Psi_*$, (16*c*) admits a solution of the form

$$f(\Psi) = (\Psi^2 + \hat{\delta})(1-m)^2,$$

with $\hat{\delta}$ an arbitrary constant, while for $\Psi > \Psi_*$, the solution is given by (23) and (25). The requirements of continuity of the speed of sound c and entropy η across $\Psi = \Psi_*$ demand the continuity of $f(\Psi)$ across $\Psi = \Psi_*$; we thus have

$$B_2 = [(1-N)\hat{\delta}\Psi_*^{-2} - N][\lambda\Psi_*^{(1-N)}]^2 \exp\left(-\frac{\eta_0}{c_p}\right).$$

This, when substituted into (25), yields

$$f(\Psi) = \lambda^2\Psi^2\left[1 + (1-N)(\delta_* - \delta)\left(\frac{\Psi}{\Psi_*}\right)^{2(N-1)}\right], \tag{26}$$

where
$$\delta_* = \hat{\delta}\Psi_*^{-2}, \quad \delta = \frac{N}{1-N}. \tag{27a, b}$$

Thus, a solution of the system (2) can be represented in the following form:

$$t = \left[\frac{c^2}{f(\Psi)}\right]^{-1/(2k)} \tag{28a}$$

$$x = \Psi\left[\frac{c^2}{f(\Psi)}\right]^{-1/(\gamma-1)} + \left(\frac{c^2}{f(\Psi)}\right)^{-1/(2k)}\left(\frac{\beta}{k}\right), \tag{28b}$$

$$u = \frac{2cH(\Psi)+\beta(\gamma+1)}{\gamma-1}, \tag{28c}$$

$$\eta = \eta_0 + 2Nc_p \ln(\Psi), \tag{28d}$$

$$H(\Psi) = \frac{k\Psi}{[f(\Psi)]^{\frac{1}{2}}}, \tag{28e}$$

where $k = 1-m$, and $f(\Psi)$ is given by (26). From (28), we have $u = [2x/((\gamma+1)t)]+\beta$ and $\Psi = xt^{-2(\gamma+1)} - \beta t^{k-1}$, where β is an arbitrary constant.

3. The non-isentropic flow in region R_4

In this section we attempt a solution in region R_4, which satisfies on the shock boundary the RH conditions

$$\frac{u_s}{c_0} = \pi\left(\frac{2}{\gamma}\right)^{\frac{1}{2}}(2\gamma+\gamma\pi+\pi)^{-\frac{1}{2}}, \tag{29a}$$

$$\eta_s - \eta_0 = c_v \ln\left[\frac{(1+\pi)(2\gamma+\gamma\pi-\pi)^{\gamma}}{(2\gamma+\gamma\pi+\pi)^{\gamma}}\right], \tag{29b}$$

$$\frac{c_s}{c_0} = \left[\frac{(\pi+1)(2\gamma+\gamma\pi-\pi)}{(2\gamma+\gamma\pi+\pi)}\right]^{\frac{1}{2}}, \tag{29c}$$

$$\frac{U}{c_0} = \left[\frac{2\gamma+\gamma\pi+\pi}{2\gamma}\right]^{\frac{1}{2}}, \tag{29d}$$

and the kinematic condition

$$\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)_s = U, \tag{29e}$$

where U is the speed of shock propagation and $\pi = (p_s/p_0)-1$ is its (pressure) strength. The subscripts s and 0 denote respectively the values just behind and ahead of the shock.

In the limit $\pi \to 0$, (28c) must reduce to the Riemann invariant, $u - 2c(\gamma-1)^{-1} =$ const. This requires that $H = 1$ when $\pi = 0$. Thus, in the limit of a sonic discontinuity propagating into a region at rest ($u_0 = 0$, $U/c_0 = 1$, $\pi = 0$, $H = 1$), (28c) yields

$$\beta = -\frac{2c_0}{\gamma+1}.$$

On evaluation of (28*c*) on the shock, we obtain H as a function of π:

$$H_s(\pi) = \frac{(\gamma-1)(2\gamma)^{-\frac{1}{2}}\pi + (2\gamma+\gamma\pi+\pi)^{\frac{1}{2}}}{[(\pi+1)(2\gamma+\gamma\pi-\pi)]^{\frac{1}{2}}}. \tag{30}$$

Next we need to determine Ψ on the shock. To do this we differentiate (28*a*, *b*) on the shock front with respect to π and insert them in the kinematic condition (29*e*) to obtain a relation of the form

$$\frac{1}{f}\left(\frac{df}{d\pi}\right)_s = 2F(\pi), \tag{31}$$

where we have used (28*e*) and where

$$F(\pi) = \frac{\left\{\frac{2}{(\gamma-1)}(H_s\zeta-1) - \frac{U}{c_0}\right\}\frac{1}{\zeta}\frac{d\zeta}{d\pi} - \zeta\frac{dH_s}{d\pi}}{\frac{\zeta H_s}{k} - \frac{U}{c_0} - \frac{2}{(\gamma-1)}}, \tag{32}$$

$$\frac{d\zeta}{d\pi} = \frac{1}{2\zeta}\left[\frac{(2\gamma+\gamma\pi+\pi)(2\gamma\pi-2\pi+3\gamma-1)-(\gamma+1)(\pi+1)(2\gamma+\gamma\pi-\pi)}{(2\gamma+\gamma\pi+\pi)^2}\right], \tag{33}$$

$$\frac{dH_s}{d\pi} = \tfrac{1}{2}H_s\left[\frac{(\gamma-1)(2/\gamma)^{\frac{1}{2}}+(\gamma+1)(2\gamma+\gamma\pi+\pi)^{-\frac{1}{2}}}{(\gamma-1)(2\gamma)^{-\frac{1}{2}}\pi+(2\gamma+\gamma\pi+\pi)^{\frac{1}{2}}} - \frac{(\gamma-1)}{(2\gamma+\gamma\pi-\pi)} - \frac{1}{(\pi+1)}\right]. \tag{34}$$

$\zeta = c_s/c_0$, H_s and U/c_0 are (known) functions of π (see (29*c*, *d*) and (30)).

Also, (26), when evaluated on the shock, yields

$$\left(\frac{1}{f}\frac{df}{d\pi}\right)_s = \frac{2}{H_s}\frac{dH_s}{d\pi}\left\{\delta + \frac{(\Psi_s/\Psi_*)^{2(1-N)}}{(1-N)^2(\delta_*-\delta)}\right\}. \tag{35}$$

On comparing (31) and (35), we obtain the following form of Ψ on the shock front;

$$\frac{\Psi_s}{\Psi_*} = \left[\frac{(\bar{F}-\delta}{(\bar{F}_1-\delta)}\right]^{1/[2(1-N)]}, \tag{36a}$$

where $\bar{F} = FH_s^{-1}(dH_s/d\pi)^{-1}$, $\bar{F}_1 = \bar{F}(\pi_1)$ and π_1 is the initial strength of the decaying shock.

Equation (36*a*) gives U/c_0 as a function of Ψ_s/Ψ_*:

$$\frac{U}{c_0} = \frac{2}{(\gamma-1)}\left\{\left[H_s\frac{d\zeta}{d\pi}+\tfrac{1}{2}(\gamma+1)\left(A-\frac{\gamma-1}{\gamma+1}\right)\zeta\frac{dH_s}{d\pi}\right]\left[\frac{1}{\zeta}\frac{d\zeta}{d\pi}+\frac{A}{H_s}\frac{dH_s}{d\pi}\right]^{-1}-1\right\}, \tag{36b}$$

where
$$A = (\delta-\bar{F}_1)\left(\frac{\Psi_s}{\Psi_*}\right)^{2(1-N)} - \delta.$$

Further, on eliminating $f(\Psi)$ from (28*a*, *e*), we obtain the following relation in the region behind the decaying shock:

$$\frac{\Psi}{\Psi_*} = \frac{\Gamma c}{kc_0}H(\Psi)\tau^k, \tag{37}$$

where $\tau = t/t_*$ is a dimensionless time variable and $\Gamma = c_0 t_*^k/\Psi_*$ is a dimensionless parameter. t_* is the time instant when the incident simple wave first strikes the uniform shock of strength π_1.

Now in order that the solution describing the non-isentropic flow behind a decaying shock may satisfy the boundary conditions (29*a–d*) and (29*e*), we require that the two expressions for Ψ/Ψ_* given by (36) and (37) match at the shock boundary. We thus arrive at the shock propagation law

$$\tau_s = \left(\frac{\zeta_1 H_1}{\zeta H_s}\right)^{\frac{1}{k}} \left\{\frac{(1-N)\bar{F}-N}{(1-N)\bar{F}_1-N}\right\}^{1/[2k(1-N)]}, \tag{38}$$

where $\zeta_1 = \zeta(\pi_1)$, $H_1 = H_s(\pi_1)$. In deriving (38), we have used the relation

$$\Gamma = \frac{k}{\zeta_1 H_1}. \tag{39}$$

This relation follows from (37), if we put $\Psi_s = \Psi_*$ and $t_s = t_*$ at the point of initial decay.

The interaction problem being discussed here arises from a certain piston motion. The piston starting with a uniform velocity u_p gives rise to a shock with constant speed U_1, and then stops. The initial shock strength π_1 is therefore uniquely determined from the piston Mach number $M_p = u_p/c_0$:

$$\pi_1 = \tfrac{1}{4}\gamma(\gamma+1)M_p^2 + [\{\tfrac{1}{4}\gamma(\gamma+1)M_p^2\}^2 + \gamma^2 M_p^2]^{\frac{1}{2}}.$$

Here, c_0 is the sound speed in the undisturbed gas. The other parameters of the problem, namely $\zeta_1 = c_s(\pi_1)/c_0$, $H_1 = H_s(\pi_1)$, $\eta_1 = \eta_s(\pi_1)$ and $\Psi_* = \exp[(\eta_1-\eta_0)/2Nc_p]$, can therefore be expressed in terms of the piston Mach number M_p.

The values of constants N and δ are determined as follows. Eliminating $f(\psi)$ from (26) and (28*e*), and substituting $\Psi_s = \Psi_*$, $t_s = t_*$ in the resulting equation, we get

$$\delta_* = \frac{1-H_1^2}{H_1^2}. \tag{40}$$

Evaluation of (28*d*) on the shock provides an expression for Ψ_s/Ψ_* which combined with (36) yields the following equation for δ:

$$\eta_2 - \eta_1 = \delta c_p \ln\left[\frac{\bar{F}_2-\delta}{\bar{F}_1-\delta}\right], \tag{41}$$

where $\eta_2 = \eta_s(\pi_2)$, $\bar{F}_2 = \bar{F}(\pi_2)$ and $\eta_1 = \eta_s(\pi_1)$. Here, π_1 and π_2 are the initial and final strengths of the decaying shock under investigation, and η_s is given by (29*b*). It may be noted that the final strength π_2 may be taken to be any real positive number as small as we please.

Equations (28*a*) and (28*b*), when evaluated on the shock, yield the shock path

$$X_s = \frac{1}{\Gamma}(\Psi_s/\Psi_*)\,\tau_s^{2/(\gamma+1)} - \frac{2\tau_s}{(\gamma-1)}, \tag{42}$$

where X_s and τ_s are the dimensionless coordinates of the shock,

$$X_s = \frac{x_s}{c_0 t_*}, \quad \tau_s = \frac{t_s}{t_*}.$$

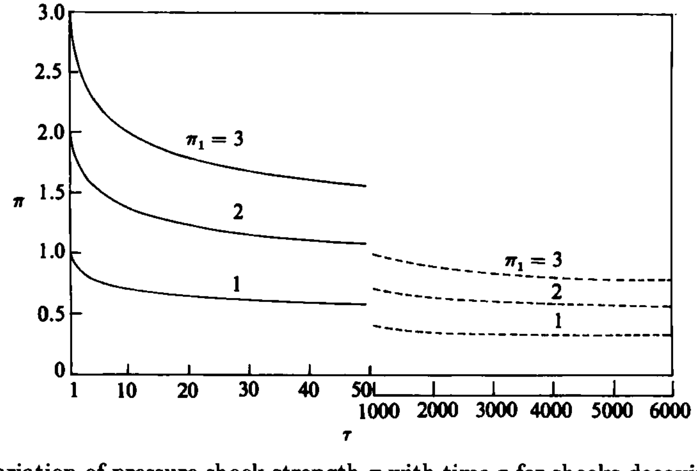

FIGURE 2. Variation of pressure shock strength π with time τ for shocks decaying from the initial strength π_1 with $\gamma = 1.4$. The decay behaviour for small (solid lines) and large (dashed lines) time limits is shown separately.

It now follows that the equation of the particle path, $\Psi_s = \Psi_*$ separating regions R_3 and R_4, is

$$X = \frac{1}{\Gamma}\tau^{2/(\gamma+1)} - \frac{2\tau}{(\gamma-1)}, \tag{43}$$

where $X = x/(c_0 t_*)$ and $\tau = t/t_*$.

In view of the foregoing results, the solution in the non-isentropic region R_4, satisfying the boundary conditions (29c, d) and (29e) can be written as

$$\frac{u}{c_0} = \frac{2}{(\gamma-1)}\left\{\frac{k}{\Gamma}\frac{\Psi}{\Psi_*}\tau^{-\kappa} - 1\right\}, \tag{44a}$$

$$\frac{c}{c_0} = \frac{k\tau^{-k}(\Psi/\Psi_*)}{\Gamma(1-N)^{\frac{1}{2}}}\left\{1 + (1-N)(\delta_* - \delta)\left(\frac{\Psi}{\Psi_*}\right)^{2N-2}\right\}^{\frac{1}{2}}, \tag{44b}$$

$$\frac{\eta - \eta_0}{c_p} = \left[\frac{\eta_1 - \eta_0}{c_p}\right] + 2N\ln\left(\frac{\Psi}{\Psi_*}\right), \tag{44c}$$

where $\Psi/\Psi_* = \Gamma\tau^{-2/(\gamma+1)}[X + 2\tau/(\gamma-1)]$, and the values of k, Γ, δ_*, δ and N are as in (28), (39), (40), (41) and (27), respectively; on the shock, Ψ/Ψ_* and τ as functions of π are given by (35) and (36), respectively.

Figures 2–6 illustrate certain features of the decay of shocks of different initial strengths π_1 for $\gamma = \frac{7}{5}$. The variation of pressure at the decaying shocks which start with the initial strengths $\pi_1 = 3$, 2 and 1, respectively, is computed from (38), and the results are shown in figure 2. The value of δ, and hence of N, is computed from (41) using the Newton–Raphson iterative method. It is evident from table 1 that the two forms of H, the first designated H^{I} and given by (30), and the second designated H^{II} and given by (28e) with Ψ_s/Ψ_* the same as in (36), remain close to one another. The results are probably accurate to better than 1 %. This is in contrast to the results obtained by Ardavan-Rhad (1970), who had a 5 % error for shocks with initial strength $\pi_1 = 3$, and much more for stronger shocks, as is evident from table 1.

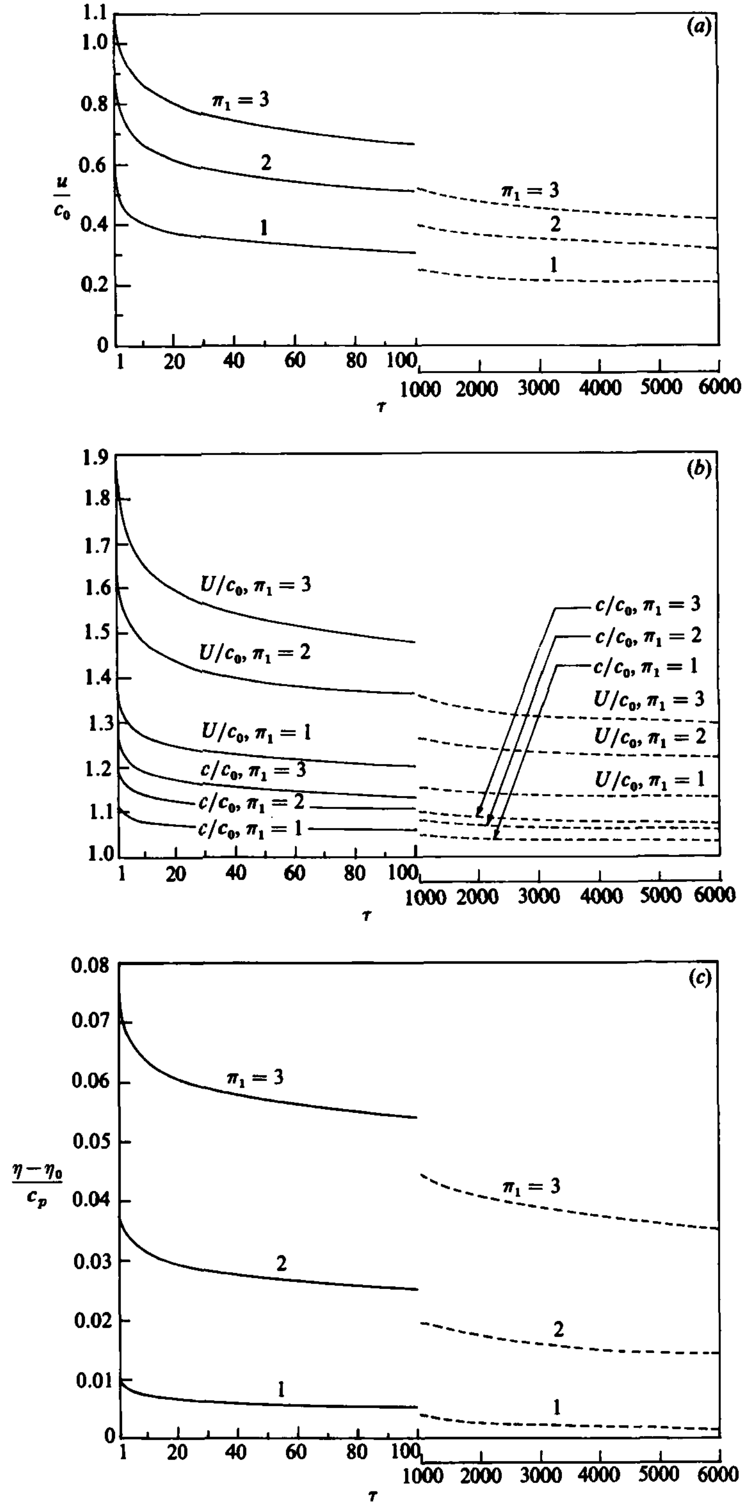

FIGURE 3. Variation of (*a*) particle velocity u/c_0, (*b*) shock speed U/c_0 and sound speed c/c_0, and (*c*) entropy at the shock with time for shocks decaying from the initial strength π_1 with $\gamma = 1.4$. The decay behaviour for small (solid lines) and large (dashed lines) time limits is shown separately.

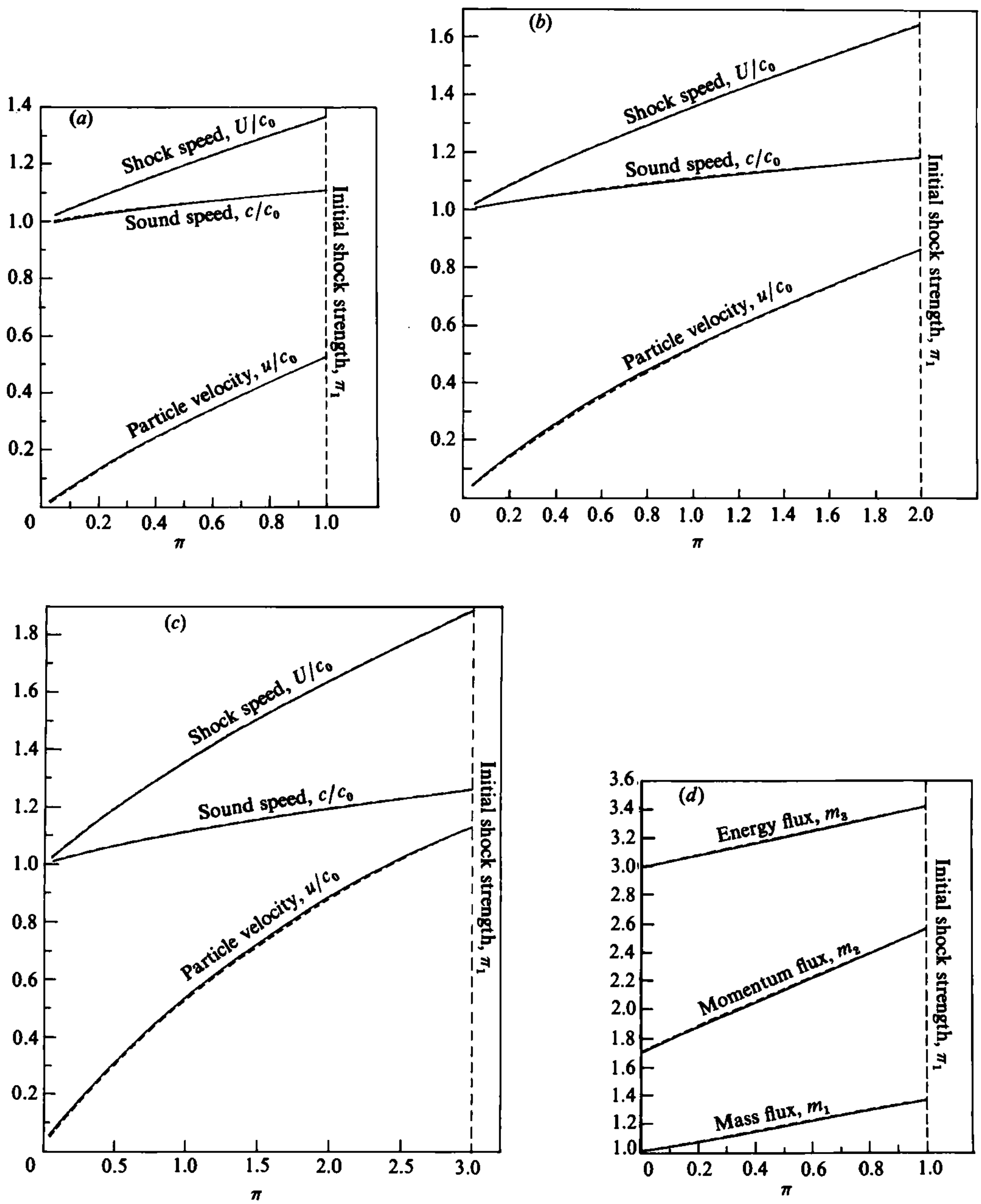

FIGURE 4 (*a–d*). For caption see facing page.

Indeed, the growing error in the approximation of the arbitrary function '*h*' of his solution would render the flow quantities incompatible with the RH conditions on the shock.

The computation of flow quantities from the solutions (36*b*) and (44) is carried out for a shock decaying from the initial strength π_1. The results are plotted in figure 3 (*a–c*), which depicts the decay of flow quantities for small and large times. The most significant behaviour of the flow variables occurs for $\tau \sim 1$, when the expansion front

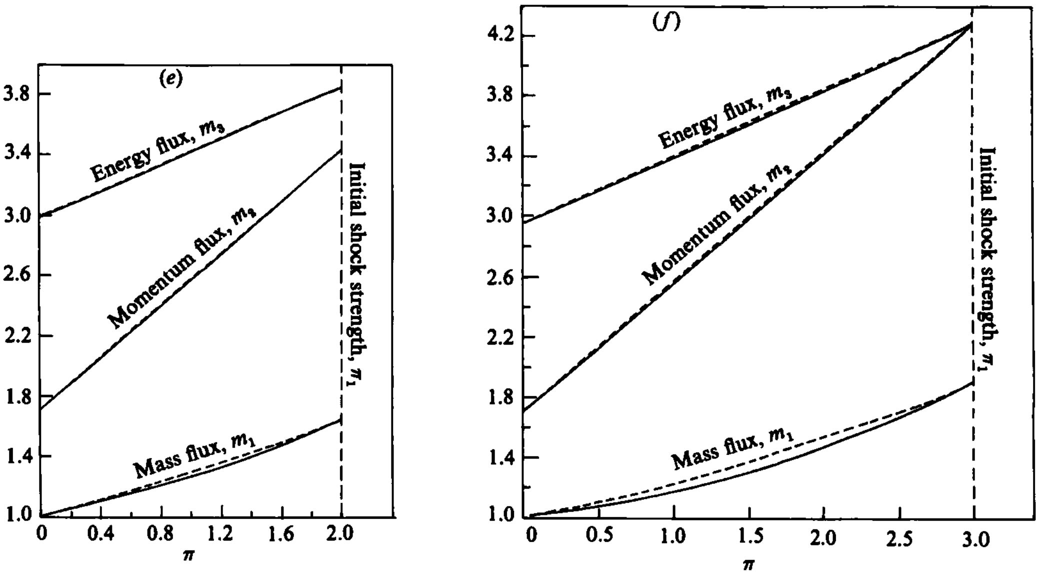

FIGURE 4. Agreement of the solution (solid lines) of the non-isentropic region with the RH conditions (dashed lines) for a shock decaying from the initial strengths (*a*) $\pi_1 = 1$, (*b*) $\pi_1 = 2$, (*c*) $\pi_1 = 3$. $\gamma = 1.4$. Physically conserved quantities from the solution (solid lines) and from the RH conditions (dashed lines) for a shock decaying from the initial strength (*d*) $\pi_1 = 1$, (*e*) $\pi_1 = 2$, (*f*) $\pi_1 = 3$. $\gamma = 1.4$.

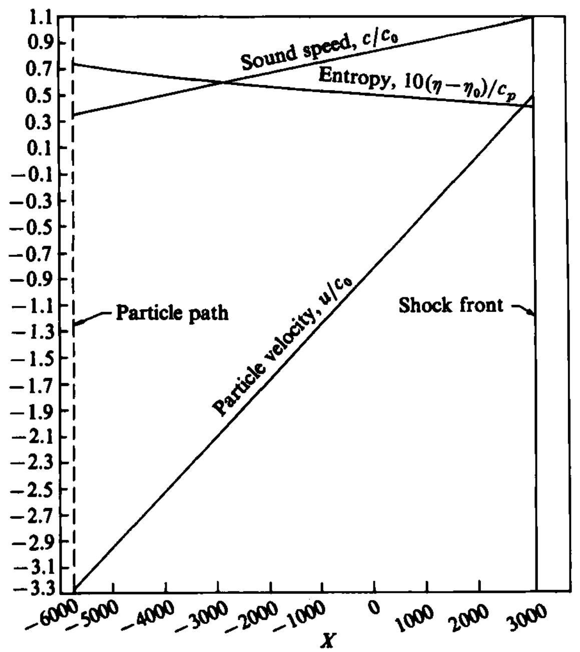

FIGURE 5. Distribution of sound speed, entropy and particle velocity in the non-isentropic flow region R_4 behind a shock decaying from the initial strength $\pi_1 = 3$ at a given time $\tau = 0.1964123 \times 10^4$.

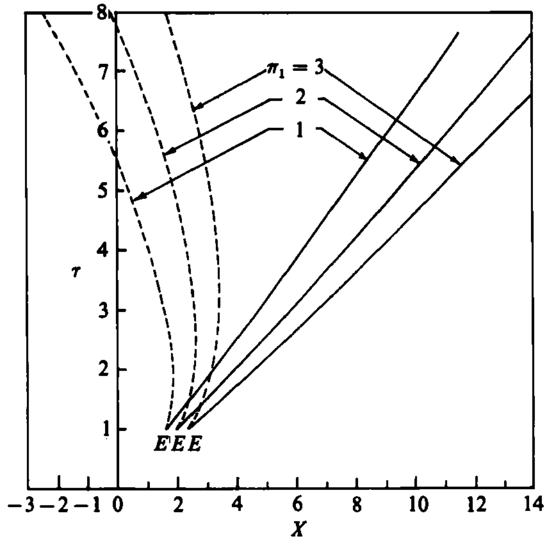

FIGURE 6. Non-uniform shock (——) decaying from the initial strength π_1 and the corresponding particle path (----) separating the non-isentropic region R_4 from the general wave region R_3. E refers to the point on the uniform shock where the leading rarefaction wave strikes the shock. Here, $\gamma = 1.4$.

strikes the shock. For a shock decaying from the initial strength π_1 the analytical expressions of the flow variables for $\tau \sim 1$ can easily be obtained from the solution

$$\frac{U}{c_0} = \left(\frac{2\gamma+\gamma\pi_1+\pi_1}{2\gamma}\right)^{\frac{1}{2}} + \frac{(\gamma-1)(1-N-H_1^2)}{2\gamma D_3}\epsilon + O(\epsilon^2),$$

$$\frac{u}{c_0} = \frac{2(\zeta_1 H_1 - 1)}{(\gamma-1)} + \frac{2(1-N-H_1^2)(D_1+D_2)}{(\gamma+1)D_3}\epsilon + O(\epsilon^2),$$

$$\frac{c}{c_0} = \zeta_1 + \frac{(\gamma-1)(1-N-H_1^2)\zeta_1 D_1}{(\gamma+1)D_3}\epsilon + O(\epsilon^2),$$

$$\frac{\eta-\eta_0}{c_p} = \frac{\eta_1-\eta_0}{c_p} + \frac{2N(\gamma-1)D_2}{(\gamma+1)D_3}\epsilon + O(\epsilon^2),$$

where

$$\epsilon = \tau - 1, \quad D_1 = \zeta_1^{-1}\left(\frac{d\zeta}{d\pi}\right)_{\pi=\pi_1}, \quad D_2 = H_1^{-1}\left(\frac{dH_s}{d\pi}\right)_{\pi=\pi_1}, \quad D_3 = (N+H_1^2)(D_1+D_2)-D_1.$$

The constants ζ_1, H_1, η_1 and N have been defined earlier. This approximate form of the solution, for $\tau \sim 1$, agrees closely with the corresponding solution depicted in figure 3(*a–c*).

The shock speed, particle velocity and sound speed are computed from (36*b*), (44*a*) and (44*b*) for $\pi_1 = 3, 2, 1$ and the results are shown in figure 4(*a–c*). These computed values from the solution (solid lines) are compared with the corresponding values from the RH conditions (dashed lines). The shock speed and the sound speed computed from the solution are virtually indistinguishable from those computed from the kinematic condition and the RH condition, even though they have different

π	H^{I}	H^{II}	Value of $h(\pi)$ in Ardavan-Rhad, which corresponds to the present value H^{II}	% Error in the present approximation	% Error in the approximation of Ardavan-Rhad
0	1	1	1	0	0
1	0.9959	0.9999	1.0028	0.4	0.7
2	0.9843	0.9883	1.0096	0.4	3.0
3	0.9698	0.9738	1.0168	0.4	5.0
4	0.9546	0.9572	1.0233	0.3	7.0
5	0.9396	0.9401	1.0290	0.1	9.0
6	0.9251	0.9237	—	0.1	—
7	0.9114	0.9086	—	0.3	—
8	0.8985	0.8960	—	0.2	—
9	0.8864	0.8841	—	0.2	—
10	0.8749	0.8749	—	0.0	—

TABLE 1. Comparison of the present approximation with that of Ardavan-Rhad (1970)

analytical forms. However, the particle velocity and the entropy show a small discrepancy. For further comparison, we examined the validity of the conservation laws of mass, momentum and energy at the shock front. Let m_1, m_2 and m_2 be the dimensionless mass, momentum and energy fluxes, respectively, at the rear of shock front; we then have

$$m_1 = \exp\left[-\frac{\gamma(\eta-\eta_0)}{(\gamma-1)c_p}\right]\left(\frac{U}{c_0}-\frac{u}{c_0}\right)\left(\frac{c}{c_0}\right)^{2/(\gamma-1)},$$

$$m_2 = \frac{U}{c_0}\left\{\left(\frac{U}{c_0}-\frac{u}{c_0}\right)+\frac{1}{\gamma}\left(\frac{c}{c_0}\right)^2\left(\frac{U}{c_0}-\frac{u}{c_0}\right)^{-1}\right\},$$

$$m_3 = \frac{1}{2}\left(\frac{U}{c_0}-\frac{u}{c_0}\right)^2+\frac{1}{(\gamma-1)}\left(\frac{c}{c_0}\right)^2.$$

m_1, m_2 and m_3 were computed from the analytic solution at the shock front as well as from the RH conditions for $\pi_1 = 1, 2, 3$. The results are plotted in figure 4(*d–f*), where solid lines (dashed lines) correspond to the results obtained from the solution (RH conditions). The comparison shows that the momentum and energy fluxes are in very good agreement but the mass flux suffers a small discrepancy. The present solution may, therefore, be interpreted as one for which the shock front exhibits a sink-like property with regard to the physically conserved quantity m_1 – the mass flux. The distribution of particle velocity, sound speed and entropy, at a given time, in region R_4 behind the decaying shock starting with initial strength $\pi_1 = 3$ is computed from (44) and the results are shown in figure 5. The shock and particle trajectories for shocks decaying from initial strengths $\pi_1 = 3$, 2 and 1 are computed from (42) and (43), and are shown in figure 6.

In the limit of a very weak shock ($\pi \ll 1$), (31), on expanding the numerator and denominator of F in powers of π, yields

$$\left(\frac{1}{f}\frac{df}{d\pi}\right)_s \approx -\frac{4k}{\pi},$$

which, on integration and use of (28*a*) and RH conditions, yields the limiting law of propagation for a weak shock, $\pi \sim t^{-\frac{1}{2}}$; this is in full agreement with the Friedrichs theory for weak shocks.

4. The isentropic flow in regions R_3, R_2 and R_1

Having determined a specific solution of the non-isentropic flow in region R_4, which describes the decay of a shock of arbitrary strength, we seek the corresponding solution for flows in regions R_3, R_2 and R_1. The flow in region R_1 behind a uniform shock is a constant state. The flow properties of this region can be fully described by the RH conditions on the shock boundary.

The region R_2 is a simple wave so that

$$u - \frac{2c}{\gamma-1} = K_*, \quad u + \frac{2c}{\gamma-1} = \xi, \tag{45}$$

where K_* is a constant given by

$$K_* = \frac{2c_0}{(\gamma+1)\Gamma}\left(1-\frac{1}{H_1}\right) - \frac{2c_0}{\gamma-1},$$

and ξ is a parameter which denominates different positive characteristics, $dx/dt = u+c$. On integrating this relation, we obtain

$$x - (u+c)\,t = g(\xi), \tag{46}$$

where $g(\xi)$ is an arbitrary function of ξ.

The analytic solution for the isentropic region R_2 can be written as

$$c = k\Psi t^{-k}, \quad u = \frac{2c}{\gamma-1} + K_*, \quad \Psi = xt^{-2/(\gamma+1)} + \frac{2c_0}{\gamma-1}t^k. \tag{47}$$

Equation (47) implies that $g(\xi) = 0$ in (46), so that the region R_2 is a centred simple wave. Thus, the isentropic flow in R_2 is described by

$$\frac{c}{c_0} = \frac{\gamma-1}{\gamma+1}\left(\frac{X}{\tau} - \frac{K_*}{c_0}\right), \tag{48}$$

$$\frac{u}{c_0} = \frac{2}{\gamma+1}\frac{X}{\tau} + \frac{\gamma-1}{\gamma+1}\frac{K_*}{c_0}, \tag{49}$$

The flow in region R_3 results from the interaction of two simple waves, one oncoming incident wave and the other receding reflected wave. The flow in this region is designated a 'general wave'. It is bounded by the particle path $\Psi = \Psi_*$ given by (43), and a characteristic front $dx/dt = u - c$ emanating from the point E (see figure 1). In view of (48) and (49), this characteristic boundary is

$$\frac{X}{\tau} = A_1\tau^{-2k} + \frac{K_*}{c_0}, \tag{50}$$

where $$A_1 = \frac{\gamma - 1 + 2H_1^{-1}}{(\gamma+1)\Gamma}.$$

The analytic solution of the isentropic flow equations in region R_3 can now be expressed in the form

$$\frac{c}{c_0} = \frac{K\tau^{-k}}{H(\Psi)} \frac{\Psi}{\Psi_*}, \tag{51}$$

$$\frac{u}{c_0} = 2(\gamma+1)^{-1} \frac{\tau^{-k}}{\Gamma} \frac{\Psi}{\Psi_*} - 2(\gamma-1)^{-1}, \tag{52}$$

where $$H(\Psi) = \frac{\Psi}{\Psi_*}\left[\delta_* + \left(\frac{\Psi}{\Psi_*}\right)^2\right]^{-\frac{1}{2}},$$

$$\frac{\Psi}{\Psi_*} = \begin{cases} 1 & \text{on the boundary (43),} \\ A_1 \Gamma \tau^{-k} - (H_1^{-1} - 1)\tau^k & \text{on the boundary (50),} \\ (X\tau^{-2/(\gamma+1)} + 2(\gamma-1)^{-1}\tau^k) & \text{in region } R_3. \end{cases}$$

It can be easily verified that the solution of the flow in region R_3 matches with those in regions R_4 and R_2 on the boundaries (43) and (50), respectively.

5. Conclusions

We have found a special solution of one-dimensional gasdynamic equations which describes the interaction of a centred simple wave with a shock of arbitrary strength. The solution satisfies the basic PDE's, the kinematic shock condition and another shock condition exactly. It satisfies the two other RH conditions (29*a*, *b*) approximately. The error incurred is shown to be small even for strong shocks. The special form of the solution has linear dependence of particle velocity on the spatial coordinate (see (12)); here the additional term $A(t)$, a function of time, is important for the present problem. This class of solutions has been referred to as self-similar (in a special sense) by Pert (1980), who considered several physical examples and showed that these self-similar forms are attained in the limit of large time. The solution we have considered bears out the importance of this special class of solutions.

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